Spring 2024 - 2025, Final Examination

Ordinary Differential Equations (Intl. Session)

Zhejiang University

Course No.: MATH1137F **Exam Date:** 2025/04/12 08:00–10:00

Format: ✓ Closed-Book / Open-Book

Instructions:
- NO notes/calculators or any other form of aid is allowed.
- In order to receive credit, you must show all of your work.

Name:____________________ **Student ID:**____________________

Section	I	II	III	IV	BONUS	Total
Score						
Grader						

I. (10 pts each problem; 30 Total.)

Score

1. Find the general solution for

$$(2xy^2 - y)\,dx + (3y^3 + x)\,dy = 0, \qquad x > 0.$$

Answer space

Score

2. Find the general solution to

$$y = 2xy' - 3(y')^2$$

and determine whether it has singular solutions.

Answer space

Score

3. Solve the equation

$$y'' = 2yy'$$

with initial conditions $y(0) = 1$, $y'(0) = 2$.

Answer space

II. (10 pts each problem; 30 Total.)

Score

1. Find the general solution to

$$x^2y'' - 5xy' + 9y = x \ln x.$$

Answer space

Score

2. Find the general solution to

$$(1 - x^2)y''' + 2xy'' - 2y' = 0.$$

Answer space

Score

3. Find the first three nonzero terms of each of the two linearly independent Frobenius series solutions for the equation

$$2x^2y'' + x(x+1)y' - 2(x+1)y = 0.$$

Answer space

III. (15 pts each problem; 30 Total.)

Score

1. Consider a system of two species with populations x and y respectively and their dynamics is governed by the following autonomous system:

$$\begin{cases} \dfrac{dx}{dt} = x^2 - 2x - xy, \\ \dfrac{dy}{dt} = y^2 - 4y + xy. \end{cases}$$

(1) Find all the critical points of the system.

(2) Characterize each of the critical points you have found in (1) as to type (node/spiral/saddle/center/etc.) and stability (stable/unstable/asymptotically stable).

Answer space

Score

2. Find the solution to the system

$$\begin{cases} \dfrac{dx}{dt} = 3x - 2y + 15, \\ \dfrac{dy}{dt} = -x + 3y - 2z, \\ \dfrac{dz}{dt} = -y + 3z. \end{cases}$$

Answer space

IV. (10 Pts)

Score

Suppose that the functions $u(x)$ and $v(x)$ are two linearly independent solutions to the second order homogeneous, linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

in which $p(x)$ and $q(x)$ are continuous functions defined on the real line.
Prove: $u(x)$ and $v(x)$ do not have common zeroes. Namely, there does not exist an $x^* \in \mathbb{R}$ such that $u(x^*) = v(x^*) = 0$. *Hint: Prove first if* $u(x^*) = 0$, $u'(x^*) \neq 0$.

Answer space

BONUS (5 Pts)

Score

Use Feynman's integration technique to prove

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi \ln 2}{8}.$$

Answer space

Summer 2024 - 2025, Final Examination

Ordinary Differential Equations (Intl. Session)

Zhejiang University

Course No.: MATH1137F **Exam Date:** 2025/06/15 10:30–12:30

Format: ✓ Closed-Book / Open-Book

Instructions:

- NO notes/calculators or any other form of aid is allowed.
- In order to receive credit, you must show all of your work.

Name:________________________ **Student ID:**________________________

Section	I	II	III	IV	BONUS	Total
Score						
Grader						

I. (10 pts each problem; 30 Total.)

Score

1. Find the general solution for

$$\left(2x - \frac{\sin x}{y}\right) dx + \frac{x^2 - \cos y}{y} dy = 0.$$

Answer space

Score

2. Solve the equation

$$y'' = (y')^3 + y'$$

with initial conditions $y(0) = \frac{\pi}{4}$, $y'(0) = 1$.

Answer space

Score

3. Find the general solution to

$$y = x^2 + 2xy' + \frac{(y')^2}{2}$$

and determine whether it has a singular solution.

Answer space

II. (10 pts each problem; 40 Total.)

Score

1. Find the general solution to

$$x^2y'' - xy' - 3y = 2x^3, \qquad x > 0.$$

Answer space

Score

2. Find the general solution to

$$(1 - x^2)y'' - 2xy' + 2y = 0.$$

Answer space

Score

3. Find the first **two** nonzero terms of each of the two linearly independent solutions in a Frobenius or power series form for the equation

$$9x(1-x)y'' - 12y' + 4y = 0$$

near $x = 0$.

Answer space

Score

4. Find the eigenvalues, λ_k, $k = 1, 2, \ldots$, and the associated eigenfunctions, $X_k(x)$, for the boundary value problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(2\pi) = 0.$$

Answer space

III. (10 Pts)

Score

Consider a uniform chain of length L and line density ρ. As illustrated below, it is initially placed on a horizontal tabletop that is elevated at a height of $2L$ above the ground. Furthermore, one-third of the chain hangs vertically over the edge of the table, and the chain begins to slide from rest along the table edge. Assume that the gravitational acceleration is a constant g and the friction on the tabletop is proportional to the chain's velocity with proportionality constant γ.

(1) Develop an ordinary differential equation model to describe the evolution of the lower end of the chain, $x(t)$.

(2) Determine the time required for the chain to completely leave the tabletop, t^*, if the tabletop is frictionless, i.e., $\gamma = 0$.

Answer space

IV. (10 pts each problem; 20 Total.)

Score

1. Find the solution to the system

$$\begin{cases} \dfrac{dx}{dt} = 2x - 7y, \\ \dfrac{dy}{dt} = 5x + 10y + 4z, \\ \dfrac{dz}{dt} = 5y + 2z. \end{cases}$$

Answer space

Score

2. Consider a system of two species with populations x and y respectively and their dynamics is governed by the following autonomous system:

$$\begin{cases} \dfrac{dx}{dt} = x^2 - 3x - xy, \\ \dfrac{dy}{dt} = -5y + xy. \end{cases}$$

(1) Find all the critical points of the system.

(2) Characterize each of the critical points you have found in (1) as to type (node/spiral/saddle/center/etc.) and stability (stable/unstable/asymptotically stable).

Answer space

BONUS (5 Pts)

Score

Prove that if the functions $P(x,y)$ and $Q(x,y)$ satisfy the ODE

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

and the equidimensional property

$$P(tx,ty) = t^m P(x,y), \qquad Q(tx,ty) = t^m Q(x,y), \qquad t \in \mathbb{R},$$

then the differential equation has an integrating factor defined by

$$\mu(x,y) = \frac{1}{xP + yQ}.$$

Answer space